Write the answers to each ***Group*** *in a separate answer-book.*

2023

MATHEMATICS — MDC

Paper : CC-1

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Group - A

[Calculus]

(Marks : 20)

1. Answer ***any four*** questions : 2×4

(a) If $\lim\limits_{x\to 0}\dfrac{\sin 2x+p\sin x}{x^3}$ be finite, find the value of p.

(b) Evaluate $\int_0^{\pi/4}\tan^5 x\,dx$.

(c) If $y=\dfrac{x}{x+1}$, find $y_5(0)$ where y_n is the nth derivative of y w.r.t. x.

(d) If $I_n=\int_0^{\frac{\pi}{2}}x^n\sin x\,dx\,(n>1)$, prove that $I_n+n(n-1)I_{n-2}=n\left(\dfrac{\pi}{2}\right)^{n-1}$.

(e) If $y=x^{2n}$ (n : a positive integer), then show that $y_n=2^n[1.3.5...(2n-1)]x^n$.

(f) Evaluate $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$.

(g) Find the interval on which the function $f(x)=x^2e^{-x}$ is monotonically decreasing.

2. Answer ***any three*** questions :

(a) Evaluate $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$. 4

(b) Obtain a reduction formula for $\int \cos^m x \sin nx\, dx$; m, n being positive integers and deduce that

$$I_{m,m} = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots + \frac{2^m}{m}\right].$$ 2+2

(c) Show that the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the x-axis is $3\pi a^2$, which is equal to three times the area of the generating circle. 4

(d) Find the length of the perimeter of the curve $r = 2(1 - \cos\theta)$. 4

(e) Find the values of a and b if $\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1$. 4

(f) Find the volume of the solid generated by the revolution of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ about x-axis. 4

Group - B
[Geometry]
(Marks : 35)

3. Answer ***any two*** questions : 2½×2

(a) If the radius of a right circular cylinder is 5, axis passes through the point (1, 2, 3) and is parallel to the straight line $\frac{x-4}{2} = \frac{y-3}{-1} = \frac{z-2}{2}$, then find the equation of the cylinder.

(b) Find the radius of the circle given by $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ and $x + 2y + 2z = 15$.

(c) Find the angle through which the axes must be turned so that the equation $lx + my + n = 0$ $(l \neq 0)$ may reduce to the form $ax + b = 0$.

(d) Determine the nature and the length of the latus rectum of the conic whose polar equation is $\frac{2}{r} = 3 - 3\cos\theta$.

4. Answer ***any five*** questions : 6×5

(a) Find the equation of the locus of the point of intersection of two tangents to the parabola $y^2 = 4ax$ such that the chord of contact subtends a right angle at the vertex.

(b) In an ellipse the normal at an extremity of the latus rectum passes through the extremity of the minor axis. Prove that $e^4 + e^2 = 1$, where e is the eccentricity of the ellipse.

(c) Find the equation of the cylinder whose guiding curve is the ellipse $4x^2 + y^2 = 1, z = 0$ and generators are parallel to the straight line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}$.

(d) Show that the equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$ parallel to the tangent at $\theta = \alpha$ is given by $l(e^2 + 2e\cos\alpha + 1) = r(e^2 - 1)[\cos(\theta - \alpha) + e\cos\theta]$.

(e) Find the nature of the surface given by the equation $3x^2 - 2y^2 - 6x - 8y - 4z = 0$.

(f) Show that the perpendiculars from the origin on the generators of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$ lie on the surface $\left(\frac{x}{a} \pm \frac{y}{b}\right)(ax \pm by) + 2z^2 = 0$.

(g) Find the equation of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$, $x + y + z = 3$ as the great circle.

(h) Reduce the equation $x^2 - 6xy + y^2 - 4x - 4y + 12 = 0$ to its canonical form and hence determine the nature of the conic.

(i) Find the equations to the generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$ which passes through the point (1, 1, 1). Hence, find the angle between these generators.

Group - C

[Vector Analysis]

(Marks : 20)

5. Answer ***any four*** questions : 2×4

(a) Find, by vector method, the volume of the tetrahedron $ABCD$ with vertices $A(1, 1, -1)$, $B(3, -2, -2)$, $C(5, 5, 3)$ and $D(4, 3, 2)$.

(b) Find the projection of the vector $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

(c) A particle, acted on by constant force $4\hat{i} + 2\hat{j} + 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ is displaced from the point $\hat{i} + 2\hat{j}$ to $2\hat{i} - \hat{j} + 3\hat{k}$. Find the work done by the force.

(d) Find the value of the constant d such that the vectors $(2\hat{i} - \hat{j} - \hat{k})$, $(\hat{i} + 2\hat{j} - 3\hat{k})$ and $(3\hat{i} + d\hat{j} + 5\hat{k})$ are coplanar.

(e) A force $5\hat{i} + 2\hat{j} - 3\hat{k}$ is applied at the point (1, –2, 2). Find the value of the moment of the force about the origin.

(f) Show that the perpendicular distance of the point $(2\hat{i} + 3\hat{j} - \hat{k})$ from the plane $\vec{r} \cdot (4\hat{i} - 3\hat{j} + \hat{k}) = 18$ is $\frac{20}{\sqrt{26}}$ units.

(g) If $\vec{u} = t^2\hat{i} - t\hat{j} + (2t + 1)\hat{k}$ and $\vec{v} = (2t - 3)\hat{i} + \hat{j} - t\hat{k}$, then show that $\frac{d}{dt}(\vec{u}.\vec{v}) = -6$ at $t = 1$.

6. Answer ***any three*** questions :

(a) By vector method, put the equation of the plane $5x - 6y + 7z = 8$ in normal form and then find the equation of the plane passing through (2, 3, 4) and parallel to the plane $5x - 6y + 7z = 8$. 2+2

(b) Prove that a necessary and sufficient condition for the vector function $\vec{a}(t)$ to have constant magnitude is $\vec{a} . \frac{d\vec{a}}{dt} = 0$. 4

(c) If $\vec{r} = 5\cos t\,\hat{i} + 5\sin t\,\hat{j} + 7t\,\hat{k}$, then find the value of $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$. 4

(d) A rigid body is spinning with an angular velocity 3 radians per second about an axis parallel to $2\hat{i} - \hat{j} + \hat{k}$ and passing through the point $\hat{i} + 2\hat{j} - 3\hat{k}$. Find the velocity of the particle at the point $-4\hat{i} + \hat{j} + \hat{k}$. 4

(e) Show that if the straight lines $\vec{r} = \vec{a} + u\vec{\alpha}$ and $\vec{r} = \vec{b} + v\vec{\beta}$ intersect, then $\left(\vec{a} - \vec{b}\right).\vec{\alpha} \times \vec{\beta} = 0$ but $\vec{\alpha} \times \vec{\beta} \neq \vec{0}$. 4

(f) Prove that for any vector $\vec{a}, \hat{i} \times \left(\vec{a} \times \hat{i}\right) + \hat{j} \times \left(\vec{a} \times \hat{j}\right) + \hat{k} \times \left(\vec{a} \times \hat{k}\right) = 2\vec{a}$. 4